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SOLUTION OF PROBLEMS IN NUMBER FOUR.

SOLUTIONS of problems in No. 4 have been received as follows:

From R. J. Adcock, 406; Marcus Baker, 403; George Eastwood, 404; W. E. Heal, 401, 404, 406, 407, 408; William Hoover, 403; Prof. P. H. Philbrick, 401, 402; P. Richardson, 402; Prof. J. Scheffer, 401, 402, 403, 404, 406, 407; Prof. E. B. Seitz, 402, 406; M. Updegraff, 401.

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401. *By M. Updegraff, Madison, Wis.*—"If two triang's are so situated that the three lines drawn thro' their corresponding vertices meet in a point, then will the corresponding sides produced meet in three points which lie on the same straight line."

SOLUTION BY THE PROPOSER.

As the lines passing through corresponding vertices of the triangles meet in a point, the triangles may be considered as projections of sections of a triangular pyramid whose edges are the three straight lines passing through the corresponding vertices. Now if a pyramid is cut by two planes these planes will intersect in a straight line, and the intersections of these two planes by the faces of the pyramid will be the bounding lines of the triangular sections of the pyramid. But if the two cutting planes are cut by a third the two traces of this third plane must meet on the line of intersection of the first two planes. Therefore the sides of the two triangular sections of the pyramid will meet in three points which lie on the line of intersection of the two cutting planes.

SOLUTION BY PROFESSORS PHILBRICK AND SCHEFFER.

Let ABC and $A'B'C'$ represent the triangles, the lines AA' BB' and CC' meeting in P. The corresponding sides meet in D, E and F.

Since $A'D$, $A'E$ and $B'F$ are respectively transversals to the triangles PAB, PAC and PBC, we have:

$$AD\times BB'\times A'P = BD\times B'P\times AA', \tag{1}$$

$$CE\times C'P\times AA' = AE\times CC'\times A'P, \tag{2}$$

$$BF\times CC'\times B'P = CF\times C'P\times BB'. \tag{3}$$

The product of (1), (2) and (3) gives:

$$AD \times CE \times BF = BD \times AE \times CF. \tag{4}$$

Hence DF is a transversal to the triangle ABC, and therefore D, E and F are in the same straight line.

402. *By Prof. W. P. Casey.*—"Upon two sides of a triangle, describe equilateral triangles, and upon the same two sides, but in the opposite direction, describe two others, and let O, O_1 be the centres of the inscribed circles in the first pair and P, P_1 those of the second pair. It is required to prove, geometrically, that the sum of the squares of the sides of the triangle $= 3(OO_1)^2 + 3(PP_1)^2$."

SOLUTION BY PROF. E. B. SEITZ.

Let ABC be any triangle, ACD, BCE and ACF, BCG the two pairs of equilateral triangles described on the sides AC and BC. Join CO, CO_1, CP, CP_1, OP, O_1P_1, OP_1, O_1P.

Let H, K, M, N, R, S, be the middle points of AC, BC, OP_1, O_1P, OO_1, PP_1, respectively. Join HK, MR, RN, NS, SM, RS, MN.

Let $BC = a$, $AC = b$, $AB = c$. Then $CO = CP = OP = \frac{1}{3}b\sqrt{3}$, $CO_1 = CP_1 = O_1P_1 = \frac{1}{3}a\sqrt{3}$. The angles ACB and OCP_1 are equal; for $\angle ACB = ACP_1 + BCP_1$ and $\angle OCP = ACP_1 + ACO$. But BCP_1 $ACO = 30°$. We also have $CA : CB :: CO : CP_1$; $\therefore$ the triangles ACB and OCP_1 are similar, and we have $AC : OC :: AB : OP_1$, whence $OP_1 = \frac{1}{3}c\sqrt{3}$. In the same way we can prove that $O_1P = \frac{1}{3}c\sqrt{3}$,

We have from similarity of triangles $MR = NS = \frac{1}{2}O_1P_1 = \frac{1}{6}a\sqrt{3}$, $RN = SM = \frac{1}{2}OP = \frac{1}{6}b\sqrt{3}$, and $HK = \frac{1}{2}c$. Since the opposite sides of $RNSM$ are equal, it is a parallelogram, and we have

$$RS^2 + MN^2 = MR^2 + RN^2 + NS^2 + SM^2 = \tfrac{1}{6}a^2 + \tfrac{1}{6}b^2 \qquad (1)$$

Since HK and RS join the middle points of the opposite sides of the q'drilateral OO_1P_1P, we have $RS^2 + HK^2 = \frac{1}{2}OP_1^2 + \frac{1}{2}O_1P^2$, whence $RS^2 = \frac{1}{12}c^2$. Substituting this value in (1), we find

$$MN^2 = \tfrac{1}{6}a^2 + \tfrac{1}{6}b^2 - \tfrac{1}{12}c^2.$$

In the quadrilateral OO_1P_1P we have

$$OO_1^2 + PP_1^2 + OP^2 + O_1P_1^2 = OP_1^2 + O_1P^2 + 4MN^2.$$

Substituting and reducing we find

$$3(OO_1)^2 + 3(PP_1)^2 = a^2 + b^2 + c^2.$$

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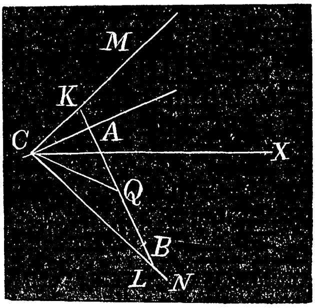
403. *By Prof. W. W. Johnson.*—"If, from the centre C of an equilateral hyperbola, CA be drawn bisecting the angle between the axis and an asymptote, and the chord AB be drawn perpendicular to CA; then $AB = 2CA$."

SOLUTION BY MARCUS BAKER, U. S. COAST SURV., LOS ANG'S, CAL.



In the annexed figure CX is the axis and CM and CN the asymptotes. CA and CQ are the bisectors of MCX and NCX. Therefore $MCA = ACX = XCQ = QCN = QLC = 22\frac{1}{2}°$;

$$AQC = ACQ = 45°;$$

and $$QKC = QCK = 67\tfrac{1}{2}°.$$

Hence the three triangles QCL, QCK and QCA are isosceles and $QL = QC = QK$ and $QA = CA$. But $AK = BL$ (property of the hyperbola) and therefore $QA = QB = AC$; $\therefore\ AB = 2AC$.

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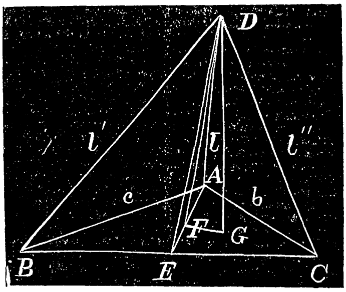
404. *By Prof. M. L. Comstock.*—"A heavy triangle ABC is suspended from a point by three strings, mutually at right angles, attached to the angular points of the triangle; if θ be the inclination of the triangle to the horizon in its position of equilibrium, then

$$\cos\theta = \frac{3}{\sqrt{(1+\sec A \sec B \sec C)}}.$$

(Todhunter's Analytical Statics, page 81.)"

SOLUTION BY PROF. J. SCHEFFER, MC SHERRYSTOWN, PA.



Let ABC represent the triangle, $DA = l$, $DB = l'$, $DC = l''$, the three strings and F, the centre of gravity of the triangle ABC.

Equilibrium will exist if F is vertically below the point D, that is, if DF is perpendicular to the horizon.

Draw DG perpendicular to the plane of ABC, then will $\angle FDG$, $= \theta$, be the angle which the $\triangle\ ABC$ makes with the horizon.

Since l, l', l'' are perpendicular to each other, we have $l^2 + l'^2 = c^2$, $l^2 + l''^2 = b^2$, $l'^2 + l''^2 = a^2$; whence

$$\left.\begin{aligned} l^2 &= \tfrac{1}{2}(b^2+c^2-a^2) = bc\cos A \\ l'^2 &= \tfrac{1}{2}(a^2+c^2-b^2) = ac\cos B \\ l''^2 &= \tfrac{1}{2}(a^2+b^2-c^2) = ab\cos C \end{aligned}\right\} \quad (1)$$

In the triangle ABC we have $b^2 + c^2 = 2.\frac{1}{4}a^2 + 2(AE)^2$, whence

$$AE^2 = \tfrac{1}{4}[2(b^2+c^2) - a^2] = m^2 \text{ say.} \quad (2)$$

Since $BE = CE$, and l', l'' are perpend. to each other, we find $DE = \frac{1}{2}a$; also

$$EF = \tfrac{1}{3}AE = \tfrac{1}{3}m. \quad (3)$$

In the triangle DEA, we have

$$\cos DEA = \frac{DE^2+AE^2-AD^2}{2DE.AE} = \frac{\frac{1}{4}a^2+m^2-l^2}{am},$$

whence we get, after substituting for m and l,

$$\cos DEA = \tfrac{1}{2}a^2. \tag{4}$$

In the triangle DEF, we have $DF^2 = DE^2+EF^2-2DE.EF.\cos DEF$. Substituting from (3) and (4), we obtain

$$DF^2 = \tfrac{1}{18}(a^2+b^2+c^2). \tag{5}$$

The volume of the tetrahedron $ABCD$ is $\frac{1}{6}ll'l''$. Denoting the area of the triangle ABC by $\triangle$, we have $\frac{1}{6}ll'l'' = \frac{1}{3}\triangle.DG$, whence

$$DG = ll'l''\div 2\triangle; \tag{6}$$

$$\therefore \cos\theta = \frac{DG}{DF} = \frac{3ll'l''}{\triangle\sqrt{2}.\sqrt{(a^2+b^2+c^2)}}. \tag{7}$$

From (1), $ll'l'' = abc\sqrt{(\cos A\cos B\cos C)}$, and since $bc = 2\triangle\div\sin A$, $ac = 2\triangle\div\sin B$, $ab = 2\triangle\div\sin C$;

$$ll'l'' = \sqrt{(8\triangle^3)}.\sqrt{(\cot A\cot B\cot C)}. \tag{8}$$

Adding the equations in (1), we get $a^2+b^2+c^2 = 2bc\cos A + 2ac\cos B + 2ab\cos C = 4\triangle(\cot A+\cot B+\cot C) = 4\triangle(\cot A\cot B\cot C + \operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C)$. Substituting in (7) and cancelling $\sqrt{(8\triangle^3)}$,

$$\cos\theta = \frac{3\sqrt{(\cot A\cot B\cot C)}}{\sqrt{(\cot A\cot B\cot C+\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C)}}.$$

Dividing numerator and denominator by $\sqrt{(\cot A\cot B\cot C)}$, we get

$$\cos\theta = \frac{3}{\sqrt{(1+\sec A+\sec B+\operatorname{cec} C)}}.$$

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405. No solution received.

406. *By William Hoover, A. M., Dayton, Ohio.*—"Find x from the eq.

$$\cot 2^{x-1}a - \cot 2^x a = \operatorname{cosec} 3a."$$

SOLUTION BY THE PROPOSER.

The given equation may be written

$$\cot\tfrac{1}{2}.2^xa - \cot 2^xa = \operatorname{cosec} 3a.$$

Put $2^xa = y$; then $\cot\frac{1}{2}y - \cot y = \operatorname{cosec} 3a$, or

$$\frac{1+\cos y}{\sin y} - \frac{\cos y}{\sin y} = \frac{1}{\sin 3a};\ \therefore \sin y = \sin 3a, \text{ or}$$

$$2^xa = 3a,\ x\log 2 = \log 3,\ x = \log 3\div\log 2.$$

407. *By Henry Heaton, Lewis, Iowa.*—"Evaluate

$$\int_0^{\frac{\pi}{2}}\left(1+\cos^4\theta\right)^{\frac{1}{2}}d\theta."$$

SOLUTION BY W. E. HEAL, MARION, INDIANA.

Let $x=\cos\theta$; the limits of x are 0 and 1.

$$\int_0^{\frac{1}{2}\pi}(1+\cos^4\theta)^{\frac{1}{2}}d\theta = \int_0^1\frac{(1+x^4)dx}{\sqrt{(1-x^2)}}.$$

$$\int\frac{(1+x^4)dx}{\sqrt{(1-x^2)}}=\int\frac{dx}{\sqrt{(1-x^2)}}+\int\frac{x^4dx}{\sqrt{(1-x^2)}}=\sin^{-1}x+\int\frac{x^4dx}{\sqrt{(1-x^2)}}$$

$$=\tfrac{1}{8}[11\sin^{-1}x-(2x^3+3x)(1-x^2)^{\frac{1}{2}}].$$

$$\therefore\int_0^1\frac{(1+x^4)dx}{\sqrt{(1-x^2)}}=\frac{1}{8}\Big[11\sin^{-1}x-(2x^3+3x)(1-x^2)^{\frac{1}{2}}\Big]_0^1=\frac{11\pi}{16}.$$

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408. *By W. E. Heal.*—Two points, one on each of two confocal ellipsoids, are said to correspond if

$$\frac{x}{a}=\frac{X}{A},\quad\frac{y}{b}=\frac{Y}{B},\quad\frac{z}{c}=\frac{Z}{C}.$$

Prove that the distance between two points, one on each of two confocal ellipsoids is equal to the distance bet. the corresp. points. (Ivory's Th.)

SOLUTION BY THE PROPOSER.

Let the equations of the ellipsoids be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{1}$$

$$\frac{X^2}{A^2}+\frac{Y^2}{B^2}+\frac{Z^2}{C^2}=1. \tag{2}$$

Since the ellipsoids are confocal

$$A^2=a^2+k^2\ldots(3);\ B^2=b^2+k^2\ldots(4);\ C^2=c^2+k^2\ldots \tag{5}$$

Let the first pair of points be

$$(x,y,z);\quad(X,Y,Z).$$

The corresponding points are

$$\left(\frac{aX}{A},\ \frac{bY}{B},\ \frac{cZ}{C}\right);\ \left(\frac{Ax}{a},\ \frac{By}{b},\ \frac{Cz}{c}\right).$$

Let $D=$ the distance between the first pair of points, and $d=$ the dist. between corresponding points;

$$D^2=(x-X)^2+(y-Y)^2+(z-Z)^2=(x^2+y^2+z^2)$$
$$-2(xX+yY+zZ)+(X^2+Y^2+Z^2). \tag{6}$$

$$d^2 = \left(\frac{Ax}{a} - \frac{aX}{A}\right)^2 + \left(\frac{By}{b} - \frac{bY}{B}\right)^2 + \left(\frac{Cz}{c} - \frac{cZ}{C}\right)^2$$

$$= \frac{A^2x^2}{a^2} + \frac{B^2y^2}{b^2} + \frac{C^2z^2}{c^2} - 2(xX+yY+zZ) + \frac{a^2X^2}{A^2} + \frac{b^2Y^2}{B^2} + \frac{c^2Z^2}{C^2}. \quad (7)$$

From (1) and (2)

$$k^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) - k^2\left(\frac{X^2}{A^2} + \frac{Y^2}{B^2} + \frac{Z^2}{C^2}\right) = 0. \quad (8)$$

Adding (7) and (8), regarding (3), (4) and (5),

$$d^2 = (x^2+y^2+z^2) - 2(xX+yY+zZ) + (X^2+Y^2+Z^2). \quad (9)$$

$$\therefore d = D.$$

PROBLEMS.

409. *By David Trowbridge, A. M., Waterburgh, N. Y.*—If in any triangle ABC, squares be described on the three sides, and the vertices of the squares be joined by the three straight lines a, b, c; show that

$$a^2 + b^2 + c^2 = 3(AB^2 + BC^2 + CA^2).$$

410. *By Prof. J. Scheffer.*—A cone with circular base is cut by a parabolic plane which passes through the centre of the base; to find the position of the centre of gravity of both portions of the cone.

411. *By Alex. S. Christie, U. S. Coast Survey.*—Sum the series

$$1 - \frac{n}{1}\,\frac{1}{3} + \frac{n(n-1)}{2!}\,\frac{1}{5} - \frac{n(n-1)(n-2)}{3!}\,\frac{1}{7} + \text{\&c.},$$

for positive values of n.

412. *By Prof. L. G. Barbour.*—Show that in any hexaedron bounded by quadrilaterals, the three lines respectively connecting the mean points of opposite (non-contiguous) faces, mutually bisect each other.

413. *By William Hoover, A. M.*—A rod rests with one extremity in a smooth plane and the other against a smooth vertical wall at an inclination a to the horizon. If it then slips down, show that it will leave the wall when its inclination is $\sin^{-1}(\frac{2}{3}\sin a)$.

414. Sum the series, $\sec\theta + \sec\frac{1}{2}\theta + \sec\frac{1}{4}\theta + \sec\frac{1}{8}\theta + \ldots + \sec\frac{1}{2^n}\theta$.

415. Evaluate $$\int\frac{dx}{x - dx}.$$